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On Subclass of Uniformly Convex and Starlike functions by Fixing Second Coefficient

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Abstract

The main object of this paper is to study some properties of the class UCT(α, β), the subclass of S consisting functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0, \quad \forall n \geq 2,$$

used by Carlson and Shaffer operator. We obtain necessary and sufficient condition for a subclass of uniformly convex functions and corresponding subclass of starlike functions with fixed second coefficient defined by Carlson and Shaffer operator for the function $f(z) \in UCT(\alpha, \beta)$. Furthermore, we obtain extreme points, closure properties for $f(z) \in UCT(\alpha, \beta)$ by fixing second coefficient.

1. Introduction

Denote by S the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

that are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$ and by ST and CV the subclasses of S that are respectively, starlike and convex. Goodman [4, 5] introduced and defined the following subclasses of CV and ST.

A function $f(z)$ is uniformly convex (uniformly starlike) in U if $f(z)$ is in $CV(ST)$ and has the property that for every circular arc γ contained in U, with center ξ also in U, the arc $f(\gamma)$ is convex (starlike) with respect to $f(\xi)$. The class of uniformly convex functions is denoged by UCV and the class of uniformly starlike functions by UST. It is well known from [[9], [10]] that

$$f \in UCV \Leftrightarrow \left|\frac{zf''(z)}{f'(z)}\right| \leq Re\left\{\frac{zf''(z)}{f'(z)}\right\}.$$

In [10], Rønning introduced a new class of starlike fucntions related to UCV defined as

$$f \in S_p \Leftrightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| \leq \left\{\frac{zf'(z)}{f(z)}\right\}.$$

Note that $f(z) \in UCV \Leftrightarrow zf'(z) \in S_p$. Further, Rønning generalized the class S_p by introducing a parameter $\alpha, -1 \leq \alpha < 1$,

$$f \in S_p(\alpha) \Leftrightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| \leq Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\}.$$

Now we define the function $\phi(a, c; z)$ by

$$\phi(a, c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n, \tag{1.2}$$

for $c \neq 0, -1, -2, \cdots, a \neq -1; z \in U$ where $(\lambda)_n$ is the Pochhammer symbol defined by

$$\begin{aligned}(\lambda)_n &= \frac{\Gamma(n+\lambda)}{(\Gamma(\lambda)} \\ &= \left\{\begin{array}{ll} 1; & n = 0 \\ \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1), & n \in= \{1, 2, \cdots\} \end{array}\right\}\end{aligned} \tag{1.3}$$

Carlson and Shaffer [3] introduced a linear operator $L(a,c)$, defined by

$$\begin{aligned} L(a,c)f(z) &= \phi(a,c;z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} a_n z^n, \quad z \in U, \end{aligned} \tag{1.4}$$

where $*$ stands for the Hadamard product or convolution product of two power series

$$\varphi(z) = \sum_{n=1}^{\infty} \varphi_n z^n \quad \text{and} \quad \psi() = \sum_{n=1}^{\infty} \psi_n z^n$$

defined by

$$(\varphi * \psi)(z) = \varphi(z) * \psi(z) = \sum_{n=1}^{\infty} \varphi_n \psi_n z^n.$$

We note that $L(a,a)f(z) = f(z), L(2,1)f(z) = zf;(z), L(m+1,1)f(z) = D^m f(z)$, where $D^m f(z)$ is the Ruscheweyh derivative of $f(z)$ defined by Ruscheweyh [11] as

$$D^m f(z) = \frac{z}{(1-z)^{m+1}} * f(z), \quad m > -1, \tag{1.5}$$

which is equivalently,

$$D^m f(z) = \frac{z}{m!} \frac{d^m}{dz^m} \{z^{m-1} f(z)\}.$$

Definition 1.1 : For $\beta \geq 0, -1 \leq \alpha < 1$, we define a class $UCV(\alpha,\beta)$ subclass of S consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic criterion

$$Re \left\{ \frac{z(L(a,c)f(z))''}{(L(a,c)f(z))'} + 1 - \alpha \right\} \geq \beta \left| \frac{z(L(a,c)f(z))''}{(L(a,c)f'(z))'} \right|, \quad z \in U. \tag{1.6}$$

We also let $UCT(\alpha,\beta)$, the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad \forall\ n \geq 2. \tag{1.7}$$

The main object of this section to obtain necessary and sufficient condition for a subclass of uniformly convex functions and corresponding subclass of starlike functions with fixed second coefficient defined by Carlson and Shaffer operator for the function

$f(z) \in UCT(\alpha, \beta)$. Furthermore, we obtain extreme points, distortion bounds and closure properties for $f(z) \in UCT(\alpha, \beta)$ by fixing second coefficient.

2. The Class $UCT(\alpha, \beta)$

Firstly, we obtain necessary and sufficient condition for functions $f(z)$ in the calsses $UCV(\alpha, \beta)$.

Theorem 2.1 : A function $f(z)$ of the form (1.1) is in $UCV(\alpha, \beta)$ if

$$\sum_{n=2}^{\infty} n[n(1+\beta) - (\alpha+\beta)] \frac{(a)_{n-1}}{(c)_{n-1}} |a_n \leq 1 - \alpha, \tag{2.1}$$

$-1 \leq \alpha < 1, \beta \geq 0$.

Proof : If suffices to show that

$$\beta \left| \frac{z(L(a,c)f(z))''}{(L(a,c)f(z))'} \right| - Re \left\{ \frac{z(L(a,c)f(z))''}{(L(a,c)f(z))'} \right\} \leq 1 - \alpha.$$

We have

$$\beta \left| \frac{z(L(a,c)f(z))''}{(L(a,c)f(z))'} \right| - Re \left\{ \frac{z(L(a,c)f(z))''}{(L(a,c)f(z))'} \right\} \leq 1 - \alpha,$$
$$\leq \frac{(1+\beta) \sum_{n=2}^{\infty} n(n-1) \frac{(a)_{n-1}}{(c)_{n-1}} |a_n|}{1 - \sum_{n=2}^{\infty} n \frac{(a)_{n-1}}{(c)_{n-1}} |a_n|}.$$

The last expression is bounded above by $(1-\alpha)$ if

$$\sum_{n=2}^{\infty} n[n(1+\beta) - (\alpha+\beta)] \frac{(a)_{n-1}}{(c)_{n-1}} |a_n| \leq 1 - \alpha,$$

and hence the proof is complete.

Theorem 2.2 : a necessary and sufficient for $f(z)$ of the form (1.7) to be in the class $UCT(\alpha, \beta), -1 \leq \alpha < 1, \beta \geq 0$ is that

$$\sum_{n=2}^{\infty} n[n(1+\beta) - (\alpha+\beta)] \frac{(a)_{n-1}}{(c)_{n-1}} a_n \leq 1 - \alpha. \tag{2.2}$$

Proof : In view of Theorem 2.1, we need only to prove the necessity. If $f(z) \in UCT(\alpha,\beta)$ and z is a real then

$$Re\left\{\frac{z(L(a,c)f(z)''}{(L(a,c)f(z))'}+1-\alpha\right\} \geq \beta\left|\frac{z(L(a,c)f(z))''}{(L(a,c)f(z))'}\right|$$

which gives

$$\Leftrightarrow \frac{-\sum\limits_{n=2}^{\infty} n(n-1)\frac{(a)_{n-1}}{(c)_{n-1}}a_n z^{n-1}+(1-\alpha)\left[\sum\limits_{n=2}^{\infty} n\frac{(a)_{n-1}}{(c)_{n-1}}a_n z^{n-1}\right]}{1-\sum\limits_{n=2}^{\infty} n\frac{(a)_{n-1}}{(c)_{n-1}}a_n z^{n-1}}$$

$$\geq \beta\left|\frac{\sum\limits_{n=2}^{\infty} n(n-1)\frac{(a)_{n-1}}{(c)_{n-1}}a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty} n\frac{(a)_{n-1}}{(c)_{n-1}}a_n z^{n-1}}\right|.$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty} n[n(1+\beta)-(\alpha+\beta)]\ \frac{(a)_{n-1}}{(c)_{n-1}}a_n \leq 1-\alpha,$$

$-1 \leq \alpha < 1, \beta \geq 0$.

Corollary 2.1 : Let the function $f(z)$ defined by (1.7) be in the class $UCT(\alpha,\beta)$. Then

$$a_n \leq \frac{(1-\alpha)(c)_{n-1}}{n[n(1+\beta)-(\alpha+\beta)](a)_{n-1}}$$

Remark 2.1 : In view of Theorem 2.2, we can see that if $f(z)$ is of the form (1.7) and is in the class $UCT(\alpha,\beta)$ then

$$a_2 = \frac{(1-\alpha)(c)}{2(2+\beta-\alpha)(a)}. \tag{2.3}$$

By fixing the second coefficient, we introduce a new subclass $UCT_b(\alpha,\beta)$ of $UCT(\alpha,\beta)$ and obtain the following theorems.

Let $UCT_b(\alpha,\beta)$ denote the class of functions $f(z)$ in $UCT(\alpha,\beta)$ and be of the form

$$f(z) = z - \frac{b(1-\alpha)(c)}{2(2+\beta-\alpha)(a)}z^2 - \sum_{n=3}^{\infty} a_n z^n \quad (a_n \geq 0), 0 \leq b \leq 1. \tag{2.4}$$

Theorem 2.3 Let the function $f(z)$ defined by (2.4). Then $f(z) \in UCT_b(\alpha, \beta)$ if and only if

$$\sum_{n=3}^{\infty} n[n(1+\beta) - (\alpha+\beta)] \frac{(a)_{n-1}}{(c)_{n-1}} a_n \leq (1-b)(1-\alpha) \tag{2.5}$$

$-1 \leq \alpha < 1, \beta > 0.$

Proof : Substituting

$$a_2 = \frac{b(1-\alpha)(c)}{2(2+\beta-\alpha)(a)}, \quad 0 \leq b \leq 1$$

in (2.2), we obtain

$$2(2+\beta-\alpha)\frac{(a)}{(c)}a_2 + \sum_{n=3}^{\infty} n[n(1+\beta) - (\alpha+\beta)]$$
$$\times \frac{(a)_{n-1}}{(c)_{n-1}} a_n \leq 1-\alpha$$

which gives

$$\sum_{n=3}^{\infty} n[n(1+\beta) - (\alpha+\beta)] \frac{(a)_{n-1}}{(c)_{n-1}} a_n \leq (1-b)(1-\alpha)$$

which is the desired result.

Corollary 2.2 : Let the function $f(z)$ defined by (2.4) be in the class $UCT_b(\alpha, \beta)$. Then

$$a_n \leq \frac{(1-b)(1-\alpha)(c)_{n-1}}{n[n(1+\beta) - (\alpha+\beta)](a)_{n-1}}, \tag{2.6}$$

$n \geq 3, -1 \leq \alpha < 1, \beta \geq 0.$

Theorem 2.4 : The class $UCT_b(\alpha, \beta)$ is closed under convex linear combination.

Proof : Let the function $f(z)$ be defined by (2.4) and $g(z)$ defined by

$$g(z) = z - \frac{b(1-\alpha)(c)}{2(2+b-\alpha)(a)} z^2 - \sum_{n=3}^{\infty} d_n z^n, \tag{2.7}$$

where $d_n \geq 0$ and $0 \leq b \leq 1$.

Assuming that $f(z)$ and $g(z)$ are in the class $UCT_b(\alpha,\beta)$, it is sufficient to prove that the function $H(z)$ defined by

$$H(z = \lambda f(z) + (1-\lambda)g(z), \quad 0 \leq \lambda \leq 1 \tag{2.8}$$

is also in the class $UCT_b(\alpha,\beta)$.

Since

$$\begin{aligned} H(z) &= z - \frac{b(1-\alpha)(c)}{2(2+\beta-\alpha)(a)}z^2 \\ &\quad - \sum_{n=3}^{\infty}\{\lambda_n + (1-\lambda)d_n\}z^n, \end{aligned} \tag{2.9}$$

$a_n \geq 0, d_n \geq 0, 0 \leq b \leq 1$, we observe that

$$\begin{aligned} &\sum_{n=3}^{\infty} n[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}\{\lambda a_n + (1-\lambda)d_n\} \\ &\leq (1-b)(1-\alpha) \end{aligned} \tag{2.10}$$

which is, in view of Theorem 2.3, implies that $H(z) \in UCT_b(\alpha,\beta)$.

This completes the proof of the theorem.

Theorem 2.5 : Let the functions

$$f_j(z) = z - \frac{b(1-\alpha)(c)}{2(2+\beta-\alpha)(a)}z^2 - \sum_{n=3}^{\infty} a_{n,j}z^n, \tag{2.11}$$

$a_{n,j} \geq 0$ be in the class $UCT_b(\alpha,\beta)$ for every j $(j = 1,2,3,\cdots,m)$. Then the function $F(z)$ defined by

$$F(z) = \sum_{j=1}^{m} \mu_j f_j(z), \tag{2.12}$$

is also in the class $UCT_b(\alpha,\beta)$, where

$$\sum_{j=1}^{\infty} \mu_j = 1. \tag{2.13}$$

Proof : Combining the definitions (2.11) and (2.12) further by (2.13) we have

$$F(z) = z - \frac{b(1-\alpha)(c)}{2(2+\beta-\alpha)(a)} \geq 2 - \sum_{n=3}^{\infty}\left(\sum_{j=1}^{m}\mu_j a_{n,j}\right) z^n. \tag{2.14}$$

Since $f_j(z) \in UCT_n(\alpha,\beta)$ for every $j = 1, 2, \cdots, m$, Theorem 2.3 yields

$$\sum_{n=3}^{\infty} n[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}} a_{n,j} \leq (1-b)(1-\alpha). \tag{2.15}$$

Thus we obtain

$$\begin{aligned}
&\sum_{n=3}^{\infty} n[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}\left(\sum_{j=1}^{m}\mu_j a_{n,j}\right)\\
&= \sum_{j=1}^{m}\left(\sum_{n=3}^{\infty} n[n(1+\beta)-(\alpha+\beta)]\right)\frac{(a)_{n-1}}{(c)_{n-1}} a_{n,j}\\
&\leq (1-b)(1-\alpha)
\end{aligned}$$

in view of Theorem 2.3. So, $F(z) \in UCT_b(\alpha,\beta)$.

Theorem 2.6 : Let

$$f_2(z) = z - \frac{b(1-\alpha)(c)}{2(2+\beta-\alpha)(a)} z^2 \tag{2.16}$$

and

$$f_n(z) = z - \frac{b(1-\alpha)(c)}{2(2+\beta-\alpha)(a)} z^2 - \frac{(1-b(1-\alpha)(c)_{n-1}}{n[n(1+\beta)-(\alpha+\beta)(a)_{n-1}} z^n \tag{2.17}$$

for $n = 3, 4, \cdots$. Then $f(z)$ is in the class $UCT_b(\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z), \tag{2.18}$$

where $\lambda_n \geq 0$ and $\sum\limits_{n=2}^{\infty} \lambda_n = 1$.

Proof : we suppose that $f(z)$ can be expressed in the form (2.18). Then we have

$$\begin{aligned} f(z) &= z - \frac{b(1-\alpha)(c)}{2(2+\beta-\alpha)(a)} z^2 \\ &\quad - \sum_{n=3}^{\infty} \lambda_n \frac{(1-b)(1-\alpha)(c)_{n-1}}{n[n(1+\beta)-(\alpha+\beta)](a)_{n-1}} z^n \\ &= z - \sum_{n=2}^{\infty} A_n z^n, \end{aligned} \tag{2.19}$$

where

$$A_2 = \frac{b(1-\alpha)(c)}{2(2+\beta-\alpha)} \tag{2.20}$$

$$A_n = \frac{\lambda_n(1-b)(1-\alpha)(c)_{n-1}}{n[n(1+\beta)-(\alpha+\beta)](a)_{n-1}}, \quad n = 3, 4, \cdots . \tag{2.21}$$

Therefore,

$$\begin{aligned} &\sum_{n=2}^{\infty} n[n(1+\beta)-(\alpha+\beta)] \frac{(a)_{n-1}}{(c)_{n-1}} A_n \\ &= b(1-\alpha) \sum_{n=3}^{\infty} \lambda_n (1-b)(1-\alpha) \\ &= (1-\alpha)[b + (1-\lambda_2)(1-b)] \\ &\leq (1-\alpha), \end{aligned} \tag{2.22}$$

It follows from Theorem 2.2 and Theorem 2.3 that $f(z)$ is in the class $UCT_b(\alpha, \beta)$. Conversely, we suppose that $f(z)$ defined by (2.4) is in the class $UCT_b(\alpha, \beta)$. Then by using (2.6), we get

$$a_n \leq \frac{(1-b)(1-\alpha)(c)_{n-1}}{n[n(1+\beta)-(\alpha+\beta)](a)_{n-1}}, \quad (n \geq 3). \tag{2.23}$$

Setting

$$\lambda_n = \frac{n[n(1+\beta)-(\alpha+\beta)](a)_{n-1}}{(1-b)(1-\alpha)(c)_{n-1}}, \quad (n \geq 3) \tag{2.24}$$

and

$$\lambda_2 = 1 - \sum_{n=3}^{\infty} \lambda_n,$$

we have (2.18). This completes the proof of Theorem 2.6.

Corolalry 2.3 : The extreme points of the class $UCT_b(\alpha,\beta)$ are functions $f_n(z), n \geq 2$ given by Theorem 2.6.

3. The Class $UCT_{b_n,k}(\alpha,\beta)$

Instead of fixing just the second coefficient, we can fix finitely many coefficients. Let $UCT_{b_n,k}(\alpha,\beta)$ be the class of functions of the form

$$f(z) = z - \sum_{n=2}^{k} \frac{b_n(1-\alpha)(c)_{n-1}}{n[n(1+\beta)-(\alpha+\beta)](a)_{n-1}} z^n - \sum_{n=k+1}^{\infty} a_n z^n; \tag{3.1}$$

where $0 \leq \sum_{n=2}^{k} b_n = b \leq 1$. Note that $UCT_{b_2,2}(\alpha,\beta) = UCT_b(\alpha,\beta)$.

Theorem 3.1 : The extreme points of the class $UCT_{b_n,k}(\alpha,\beta)$ are

$$f_k(z) = z - \sum_{n=2}^{k} \frac{b_n(1-\alpha)(c)_{n-1}}{n[n(1+\beta)-(\alpha+\beta)](a)_{n-1}} z^n$$

and

$$\begin{aligned} f(n(z) &= z - \sum_{n=2}^{\infty} \frac{b_n(1-\alpha)(c)_{n-1}}{n[n(1+\beta)-(\alpha+\beta)](a)_{n-1}} z^n \\ &\quad - \sum_{n=k+1}^{\infty} \frac{(1-b)(1-\alpha)(c)_{n-1}}{n[n(1+\beta)-(\alpha+\beta)](a)_{n-1}} z^n. \\ &= z - \sum_{n=2}^{\infty} \frac{b_n(1-\alpha)(c)_{n-1}}{n[n(1+\beta)-(\alpha+\beta)](a_{n-1}} z^n \\ &\quad - \sum_{n=k+1}^{\infty} \frac{(1-b)(1-\alpha)(c)_{n-1}}{n[n(1+\beta)-(\alpha+\beta)](a)_{n-1}} z^n. \end{aligned}$$

The details of the proof are omitted, since the characterization of the extreme points enables us to solve the standard extremal problems in the same manner as was done for $UCT_b(\alpha,\beta)$.

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